

GENERALIZED TWISTED SECTORS OF ORBIFOLDS

CARLA FARSI AND CHRISTOPHER SEATON

ABSTRACT. For a finitely generated discrete group Γ , the Γ -sectors of an orbifold Q are a disjoint union of orbifolds corresponding to homomorphisms from Γ into a groupoid presenting Q . Here, we show that the inertia orbifold and k -multi-sectors are special cases of the Γ -sectors, and that the Γ -sectors are orbifold covers of Leida's fixed-point sectors. In the case of a global quotient, we show that the Γ -sectors correspond to orbifolds considered by other authors for global quotient orbifolds as well as their direct generalization to the case of an orbifold given by a quotient by a Lie group. Furthermore, we develop a model for the Γ -sectors corresponding to a generalized loop space.

1. INTRODUCTION

In [9], the authors introduced the Γ -sectors of an orbifold in order to determine a complete obstruction to the existence of a nonvanishing vector. The definitions of these sectors was heavily motivated by several existing constructions for orbifolds by Kawasaki ([12], [13], and [14]), Chen and Ruan ([6], [19]), Bryan and Fulman ([4]), and Tamanoi ([20] and [21]).

The goal of this paper is to show explicitly how the Γ -sectors generalize these constructions. In particular, we show that the inertia orbifold corresponds to the \mathbb{Z} -sectors and the k -multi-sectors correspond to the \mathbb{F}_k -sectors where \mathbb{F}_k is the free group with k generators. The orbifolds whose Euler characteristics are considered by Bryan-Fulman and Tamanoi for global quotients correspond to the \mathbb{Z}^k -sectors and Γ -sectors, respectively, in the case that Q can be expressed as a global quotient; i.e. a quotient of a manifold by a finite group. Additionally, we show that the fixed-point sectors introduced by Leida in [15] are orbifold-covered by the Γ -sectors for an appropriate choice of Γ .

The work of Lupercio and Uribe in [16] (see also [8]) demonstrates that the inertia orbifold naturally appears when considering the loop space of an orbifold. Here, we show that the same holds true for the Γ -sectors; in particular, they appear when considering smooth maps $M_\Gamma \rightarrow Q$ where M_Γ is a smooth manifold with fundamental group Γ . This generalizes results of Tamanoi in [21], stated for global quotients in the context of orbifold bundles.

In the case that an orbifold Q is presented by a quotient M/G where M is a manifold and G is a Lie group acting locally freely, i.e. properly with discrete stabilizers, there is a very natural extension of the definition of the orbifolds considered by Bryan-Fulman and Tamanoi (see Definition 2.1). We show that this again coincides with the Γ -sectors. Note, however, that such a presentation of the Γ -sectors

2000 *Mathematics Subject Classification.* Primary 22A22, 57S15; Secondary 57S17, 58H05.

Key words and phrases. orbifold, orbifold groupoid, twisted sector, loop groupoid.

The second author was partially supported by a Rhodes College Faculty Development Endowment Grant.

leads to a different indexing of the sectors. In the case of a global quotient, the Γ -sectors are naturally indexed by G -conjugacy classes $(\phi)_\sim$ of homomorphisms $\phi : \Gamma \rightarrow G$ whose images fix a nonempty subset of M ; we use $t_{M;G}^\Gamma$ to denote the set of conjugacy classes of such homomorphisms (see Subsection 2.1). On the other hand, if \mathcal{G} is an orbifold groupoid presenting Q , then the sectors are indexed by elements of T_Q^Γ , the set of \approx -classes of elements of $\text{HOM}(\Gamma, \mathcal{G})$ or equivalently connected components of $|\mathcal{G} \times \text{HOM}(\Gamma, \mathcal{G})|$ (see [9, Subsection 2.2] or Subsection 2.2 below). The discrepancy arises from the fact that the fixed-point set of a homomorphism $\phi : \Gamma \rightarrow G$ need not be connected, and hence the Γ -sector corresponding to $(\phi)_\sim \in t_{M;G}^\Gamma$ may correspond to the disjoint union of several Γ -sectors, each corresponding to one element of T_Q^Γ (see Example 3.2).

Note that in [9], we made the requirement that our local groups act with a fixed-point set of codimension 2; however, it was noted that the construction of the Γ -sectors did not require this property. In this paper, we do not make this requirement. Moreover, while our primary interest is the case of orbifold groupoids, many of these constructions and results generalize directly to the case of orbispaces (see [5]). We mention these generalizations as they arise.

The first author would like to thank the MSRI for its hospitality during the preparation of this manuscript.

2. TWO DEFINITIONS OF THE Γ -SECTORS FOR QUOTIENT ORBIFOLDS

In [9], the Γ -sectors of a general orbifold were constructed in terms of the orbifold structure given by an orbifold groupoid \mathcal{G} ; that is, a proper, étale Lie groupoid. For background on orbifolds from this perspective, the reader is referred to [1]; see also [18] and [17]. In Subsection 2.1, we are primarily concerned with orbifolds presented as the quotient of a manifold by a Lie group. We construct the Γ -sectors directly from such a presentation. This construction was introduced by Tamanai in [20] and [21] for the case that G is finite; the definitions are unchanged for general G . In Subsection 2.2, we review the key points of the construction in [9] and give other interpretations. Note that we use slightly different notation for the Γ -sectors of a quotient orbifold to distinguish from the construction using a general orbifold groupoid; these definitions will be compared in Section 3.

2.1. Γ -Sectors of a Quotient Presentation. Let Q be an n -dimensional *quotient orbifold*. By this, we mean that Q is presented by $G \ltimes M$ where M is a smooth manifold, G is a Lie group acting smoothly on M , and $G \ltimes M$ is Morita equivalent to an orbifold groupoid, i.e. a proper étale Lie groupoid. In [2, page 536] and [1, page 57] (see also [12, page 76]), it is noted that this is the case whenever the following conditions are satisfied:

- i. the isotropy group G_x for each $x \in M$ is finite,
- ii. there is a smooth slice S_x at each $x \in M$, and
- iii. for each $x, y \in M$ with $y \notin Gx$, there are slices S_x and S_y such that $GS_x \cap GS_y = \emptyset$.

In particular, it is noted that (ii) and (iii) are automatically satisfied if G is compact. The following special cases are worth noting; occasionally, we will restrict our attention to one of these.

- If G is a finite group, then Q is a **global quotient orbifold**.

- If G is a discrete group acting properly discontinuously, then Q is a **good** orbifold (see [22, Definition 13.2.3] or [3, page 20]).

Note that we use the notation M/G to indicate the quotient space as a topological space only; the orbifold (i.e. Morita equivalence class of the groupoid $G \ltimes M$) will generally be denoted Q . In [11], the question of whether every orbifold can be expressed as a quotient is addressed. In general, this question remains unresolved.

Note that in the case of a good orbifold (including the case of a global quotient), the groupoid $G \ltimes M$ is an orbifold groupoid. On the other hand, if G is a Lie group of positive dimension, then $G \ltimes M$ is not étale, though it is Morita equivalent to an orbifold groupoid. In general, $G \ltimes M$ as well as any Morita equivalent groupoid will always be a proper foliation groupoid (see [1, pages 18 and 21] and [7] for more details).

Let Γ be a finitely generated discrete group—although many of our constructions make sense for arbitrary Γ , we are only interested in this case. If ϕ and ψ are homomorphisms from Γ to G , we say $\phi \sim \psi$ if they are pointwise conjugate; i.e. if there is a $g \in G$ such that $g\phi(\gamma)g^{-1} = \psi(\gamma)$ for each $\gamma \in \Gamma$. We let (ϕ) denote the conjugacy class of ϕ (or sometimes $(\phi)_\sim$ to distinguish from equivalence classes via other relations), and let $t_{M;G}^\Gamma$ denote the set of conjugacy classes of homomorphisms ϕ whose images have nonempty fixed-point sets in M . We let $M^{(\phi)}$ denote the fixed-point set of the image of ϕ in G and $C_G(\phi)$ is the centralizer of the image of ϕ in G .

Definition 2.1. *Let $\phi : \Gamma \rightarrow G$ be a homomorphism with $M^{(\phi)} \neq \emptyset$. Then the Γ -sector of $G \ltimes M$ associated to (ϕ) is the orbifold with presentation*

$$(M; G)_{(\phi)} := C_G(\phi) \ltimes M^{(\phi)}.$$

We let $(M; G)_\Gamma$ denote the disjoint union of the Γ -sectors,

$$(M; G)_\Gamma := \coprod_{(\phi) \in t_{M;G}^\Gamma} (M; G)_{(\phi)}.$$

If G is finite, it is obvious that each $(M; G)_{(\phi)}$ is an orbifold groupoid (i.e. a proper étale Lie groupoid). We will see in Corollary 3.3 that this is generally the case.

If $x \in M^{(\phi)} \subseteq M$, we will sometimes use the notation (x, ϕ) to distinguish between $(x, \phi) \in M^{(\phi)}$ and $(x, 1) \in M^{(1)} = M$. Hence, we use $C_G(\phi)(x, \phi)$ to denote the corresponding point in $(M; G)_{(\phi)}$.

The following lemma, whose proof is standard, ensures that the definition of $(M; G)_{(\phi)}$ does not depend on the choice of the representative of the class (ϕ) .

Lemma 2.2. *Let G be a group acting on the smooth manifold M such that $G \ltimes M$ presents a smooth orbifold and let Γ be a finitely generated discrete group. If $\phi, \psi : \Gamma \rightarrow G$ are conjugate homomorphisms with $\psi = g\phi g^{-1}$ for $g \in G$, then the map*

$$\begin{aligned} L_g : M^{(\phi)} &\longrightarrow M^{(\psi)} \\ (x, \phi) &\longmapsto (gx, g\phi g^{-1}) = (gx, \psi) \end{aligned}$$

is a $C_G(\phi)$ - $C_G(\psi)$ -equivariant diffeomorphism that induces a groupoid isomorphism between $(M; G)_{(\phi)}$ and $(M; G)_{(\psi)}$. Moreover, $\sigma|_{M^{(\phi)}} = \sigma|_{M^{(\psi)}} \circ L_g$.

Note in particular that G acts on the set $\coprod_{\phi \in \text{HOM}(\Gamma, G)} (M^{(\phi)}, \phi)$ by defining $g(x, \phi) = (gx, g\phi g^{-1})$. The following lemma introduces a different presentation for $(M; G)_\Gamma$.

Lemma 2.3. *Suppose $G \ltimes M$ presents a quotient orbifold and let Γ be a finitely generated discrete group. There is a strong equivalence*

$$G \ltimes \coprod_{\psi \in \text{HOM}(\Gamma, G)} (M^{(\psi)}, \psi) \longrightarrow (M; G)_\Gamma.$$

Hence, $G \ltimes \coprod_{\psi \in \text{HOM}(\Gamma, G)} (M^{(\psi)}, \psi)$ and $(M; G)_\Gamma$ are Morita equivalent.

By strong equivalence, we mean an equivalence of groupoids such that the map on objects is a surjective submersion; see [1, page 20]. Note that neither of the groupoids in question need be orbifold groupoids; we will see in Section 3 that they are both Morita equivalent to orbifold groupoids.

Proof. Pick $\phi \in \text{HOM}(\Gamma, G)$ with $M^{(\phi)} \neq \emptyset$. Here, we denote points in $M^{(\phi)}$ simply as x to distinguish from points in $\coprod_{\psi \in (\phi)} (M^{(\psi)}, \psi)$. For each $\psi \in (\phi)$, pick a $g_\psi \in G$ such that $g_\psi \psi g_\psi^{-1} = \phi$. We require that $g_\phi = 1$. Define the map

$$\begin{aligned} \Psi_0^\phi : \quad \coprod_{\psi \in (\phi)} (M^{(\psi)}, \psi) &\longrightarrow M^{(\phi)} \\ &: (x, \psi) \longmapsto g_\psi x. \end{aligned}$$

Similarly, as G acts on $\coprod_{\psi \in (\phi)} (M^{(\psi)}, \psi)$, define

$$\begin{aligned} \Psi_1^\phi : \quad G \times \coprod_{\psi \in (\phi)} (M^{(\psi)}, \psi) &\longrightarrow C_G(\phi) \times M^{(\phi)} \\ &: (g, (x, \psi)) \longmapsto (g(g_\psi \psi g_\psi^{-1}), g_\psi x). \end{aligned}$$

It is easy to check that Ψ_0^ϕ and Ψ_1^ϕ are smooth, and that they form the maps on objects and arrows, respectively of a groupoid homomorphism $\Psi^\phi : G \ltimes \coprod_{\psi \in (\phi)} (M^{(\psi)}, \psi) \rightarrow C_G(\phi) \ltimes M^{(\phi)}$.

As Ψ_0^ϕ is a disjoint union of diffeomorphisms, Ψ_0^ϕ is a surjective submersion. It remains to show that

$$\begin{array}{ccc} G \times \coprod_{\psi \in (\phi)} (M^{(\psi)}, \psi) & \xrightarrow{\Psi_1^\phi} & C_G(\phi) \times M^{(\phi)} \\ s_1 \times s_2 \downarrow & & \downarrow s_1 \times t_1 \\ \coprod_{\psi \in (\phi)} (M^{(\psi)}, \psi) \times \coprod_{\psi \in (\phi)} (M^{(\psi)}, \psi) & \xrightarrow{\Psi_0^\phi \times \Psi_0^\phi} & M^{(\phi)} \times M^{(\phi)} \end{array}$$

is a fibered product of manifolds. This follows from the fact that the map

$$\begin{aligned} \Phi^\phi : \quad G \times \coprod_{\psi \in (\phi)} (M^{(\psi)}, \psi) &\longrightarrow \left(\coprod_{\psi \in (\phi)} (M^{(\psi)}, \psi) \right) \times \left(\coprod_{\psi \in (\phi)} (M^{(\psi)}, \psi) \right) \times (C_G(\phi) \times M^{(\phi)}) \\ &: (h, (w, \psi)) \longmapsto ((w, \psi), (hw, h\psi h^{-1}), (g_{(h\psi h^{-1})} h g_\psi^{-1}, g_\psi w)). \end{aligned}$$

is a diffeomorphism, which is easy to verify.

With this, we need only note that $G \ltimes \coprod_{\phi \in \text{HOM}(\Gamma, G)} (M^{(\phi)}, \phi)$ admits a decomposition into disjoint groupoids

$$G \ltimes \coprod_{\phi \in \text{HOM}(\Gamma, G)} (M^{(\phi)}, \phi) = \coprod_{(\phi) \in t_{M, G}^\Gamma} G \ltimes \coprod_{\psi \in (\phi)} (M^{(\psi)}, \psi),$$

and each Ψ^ϕ maps one of these groupoids into $(M; G)_{(\phi)}$. Hence,

$$\Psi := \coprod_{(\phi) \in t_{M, G}^\Gamma} \Psi^\phi : G \ltimes \coprod_{\phi \in \text{HOM}(\Gamma, G)} (M^{(\phi)}, \phi) \longrightarrow (M; G)_\Gamma$$

is clearly surjective, and therefore is a strong equivalence. \square

Note that the maps Ψ^ϕ depend on the choice of the $g_\psi \in G$. However, it is easy to see that the induced map on orbit spaces does not depend on this choice.

Fix a homomorphism $\phi : \Gamma \rightarrow G$. Then the injection $M^{(\phi)} \hookrightarrow M$ induces a map

$$\begin{aligned} \pi_{(\phi)} : M^{(\phi)} / C_G(\phi) &\rightarrow M / G \\ : C_G(\phi)(x, \phi) &\mapsto Gx \end{aligned}$$

If $g\phi g^{-1} = \psi$ and $(x, \phi) \in M^{(\phi)}$, then the G -orbit of x in M coincides with that of the corresponding point $g(x, \psi) \in M^{(\psi)}$. Therefore, this map does not depend on the particular choice of representative from the conjugacy class (ϕ) . If $(x, \phi) \in M^{(\phi)}$, then $\sigma(x) = \pi_{(\phi)}(C_G(\phi)(x, \phi))$. In particular, $\pi_{(\phi)}(M^{(\phi)}) = \sigma(M^{(\phi)})$.

Finally, note that the map $M / C_G(\phi) \rightarrow M / G$ defined by $C_G(\phi)(x, \phi) \mapsto Gx$ is an orbifold cover by definition (see [1, Definition 2.16]). The map $\pi_{(\phi)}$ is the restriction of this orbifold cover to $M^{(\phi)} / C_G(\phi)$.

2.2. Γ -Sectors for a General Presentation. In this subsection, we review the construction of the Γ -sectors for a general orbifold Q . We state the construction in general for an arbitrary orbifold groupoid \mathcal{G} . Throughout, we use the convention that the groupoid \mathcal{G} has space of objects G_0 and space of arrows G_1 . We also let $\sigma : G_0 \rightarrow |\mathcal{G}| = Q$ denote the quotient map.

If Γ and \mathcal{G} are groupoids (with no additional hypotheses), then let $\mathcal{S}_\mathcal{G}^\Gamma$ denote the set of groupoid homomorphisms $\phi : \Gamma \rightarrow \mathcal{G}$ such that the map on objects is constant. Then \mathcal{G} acts on $\mathcal{S}_\mathcal{G}^\Gamma$ by conjugation; if $\phi_0(z) = x$ for each $z \in \Gamma_0$, then for each $g \in G_1$ with $s(g) = x$, we let $(g \cdot \phi) : \Gamma \rightarrow \mathcal{G}$ have constant map on objects with value $t(g)$ and map on arrows $(g \cdot \phi)_1(\gamma) = g\phi_1(\gamma)g^{-1}$ for each $\gamma \in \Gamma_1$.

If Γ is a group (treated as a groupoid with one unit), then every homomorphism $\Gamma \rightarrow \mathcal{G}$ is constant on objects and corresponds to choice of $x \in G_0$ and group homomorphism $\phi_x : \Gamma \rightarrow G_x$ where G_x denotes the isotropy group of x . Hence, we use ϕ_x to denote the corresponding groupoid homomorphism.

If \mathcal{G} is a topological groupoid presenting an orbispace X (see [5] or [10]), then each point $x \in G_0$ is contained in an open, connected, locally connected $U \subseteq G_0$ such that $\mathcal{G}|_U$ is isomorphic to $G_U \ltimes U$ where G_U is a topological group acting continuously on U . We give $\mathcal{S}_\mathcal{G}^\Gamma$ the weak topology induced by the maps $\beta_\mathcal{G} : \phi_x \mapsto x \in G_0$ and for each $\gamma \in \Gamma$ the evaluation $\epsilon_\gamma : \phi_x \mapsto \phi_x(\gamma) \in G_1$. It is easy to check that the \mathcal{G} -action on $\mathcal{S}_\mathcal{G}^\Gamma$ is continuous. With this, we make the following.

Definition 2.4. Let \mathcal{G} be a topological groupoid representing an orbispace X and let Γ be a finitely generated discrete group. The Γ -sector groupoid of \mathcal{G} , denoted \mathcal{G}^Γ , is the translation groupoid $\mathcal{G} \ltimes \mathcal{S}_\mathcal{G}^\Gamma$.

For each $\phi_x \in \mathcal{S}_\mathcal{G}^\Gamma$ choosing $x \in U \subseteq \mathcal{S}_\mathcal{G}^\Gamma$ as above induces an isomorphism of topological groupoids between $C_{G_U}(\phi_x) \ltimes U^{\langle \phi_x \rangle}$ and the restriction of \mathcal{G}^Γ to the connected component of $\beta_\Gamma^{-1}(U)$ containing ϕ_x . It follows that the Γ -sector groupoid represents an orbispace $|\mathcal{G}^\Gamma|$. As a set,

$$(2.1) \quad |\mathcal{G}^\Gamma| = \left\{ \left(p, (\phi_x)_{G_x} \right) : p = \mathcal{G}x \in |\mathcal{G}|, \phi_x \in \text{HOM}(\Gamma, G_x) \right\}$$

where $(\phi_x)_{G_x}$ denotes the conjugacy class of the homomorphism ϕ_x in G_x .

Now assume that \mathcal{G} is an orbifold groupoid presenting the orbifold Q and Γ is a finitely generated discrete group. Then if $\phi_x, \psi_y \in \mathcal{S}_\mathcal{G}^\Gamma$, a natural transformation from ϕ_x to ψ_y is simply a choice of an arrow $g \in G_1$ such that $s(g) = x$, $t(g) = y$, and $\psi_x(\gamma)g = g\psi_y(\gamma)$ for each $\gamma \in \Gamma$. Moreover, if $\epsilon : \mathcal{K} \rightarrow \Gamma$ is an equivalence, then ϵ is locally invertible, and $\phi_x \circ \epsilon^{-1}$ is equivalent to ϕ_x (see [1, Example 2.42]). It follows that the orbits of points in $\mathcal{S}_\mathcal{G}^\Gamma$ via the \mathcal{G} -action correspond exactly to groupoid morphisms from Γ to \mathcal{G} .

For each point $p \in Q$ corresponding to the orbit of $x \in G_0$, there is a linear orbifold chart $\{V_x, G_x, \pi_x\}$ for Q at x . By this, we mean that $V_x \subseteq G_0$ is diffeomorphic to \mathbb{R}^n with x corresponding to the origin, G_x acts linearly on V_x , and there is a groupoid isomorphism between $\mathcal{G}|_{V_x}$ and $G_x \ltimes V_x$. We let $\xi_x : (s, t)^{-1}(V_x \times V_x) \rightarrow G_x$ denote the identification given by this isomorphism and $\xi_x^y = (\xi_x)|_{G_y} : G_y \rightarrow G_x$ the injective homomorphism given by restriction to G_y for each $y \in V_x$.

In this case, $\mathcal{S}_\mathcal{G}^\Gamma$ is a smooth manifold (with connected components of different dimensions) and that the \mathcal{G} -action is smooth. Hence the translation groupoid $\mathcal{G}^\Gamma = \mathcal{G} \ltimes \mathcal{S}_\mathcal{G}^\Gamma$ is an orbifold groupoid, defining an orbifold structure for the Γ -sectors of Q , denoted \tilde{Q}_Γ . For each $\phi_x \in \mathcal{S}_\mathcal{G}^\Gamma$, there is a diffeomorphism κ_{ϕ_x} of $V_x^{\langle \phi_x \rangle}$ onto a neighborhood of ϕ_x in $\mathcal{S}_\mathcal{G}^\Gamma$ forming a manifold chart. Identifying $V_x^{\langle \phi_x \rangle}$ with its image via κ_{ϕ_x} , $\{V_x^{\langle \phi_x \rangle}, C_{G_x}(\phi_x), \pi_x^{\phi_x}\}$ forms a linear orbifold chart for \tilde{Q}_Γ at ϕ_x .

Within a linear chart $\{V_x, G_x, \pi_x\}$ at x with $y \in V_x$, we say that ϕ_x locally covers ψ_y (written $\phi_x \overset{\text{loc}}{\curvearrowright} \psi_y$) if there is a $g \in G_x$ such that $g[(\xi_x^y \circ \psi_y)(\gamma)]g^{-1} = \phi_x(\gamma)$. Then by [9, Lemma 2.7], there is a $\psi_{y'} \in \mathcal{G}\psi_y$ such that $\xi_x^y \circ \psi_{y'} = \phi_x$. Extending this to an equivalence relation on $\mathcal{S}_\mathcal{G}^\Gamma$, we say that $\phi_x \approx \psi_y$ if there is a finite sequence of local coverings (in either direction) connecting an element of $\mathcal{G}\phi_x$ to $\mathcal{G}\psi_y$. We let $(\phi)_\approx$ denote the \approx -class of ϕ and T_Q^Γ denote the set of \approx -classes in $\mathcal{S}_\mathcal{G}^\Gamma$; when there is no risk of confusion, we simply denote the \approx -class of ϕ by (ϕ) . The \approx -classes in $\mathcal{S}_\mathcal{G}^\Gamma$ correspond exactly to the connected components of \tilde{Q}_Γ , so for each $(\phi) \in T_Q^\Gamma$, we let $\tilde{Q}_{(\phi)}$ denote the connected component consisting of \mathcal{G} -orbits of elements of (ϕ) and refer to $\tilde{Q}_{(\phi)}$ as the Γ -sector associated to (ϕ) .

Note that in [9, Lemma 2.5], it was shown that a strong equivalence between orbifold groupoids induces a strong equivalence between their associated groupoids of Γ -sectors. Here, we will be interested in foliation groupoids that are not necessarily étale. Hence, we note the following.

Lemma 2.5. Suppose \mathcal{G} and \mathcal{G}' are Morita equivalent orbifold groupoids. Then they are Morita equivalent via orbifold groupoids; i.e. there is an orbifold groupoid

\mathcal{H} and strong equivalences

$$\mathcal{G} \xleftarrow{e} \mathcal{H} \xrightarrow{\prime} \mathcal{G}'$$

Of course, such an \mathcal{H} always exists, and it is always a proper foliation groupoid. The point of this lemma is that \mathcal{H} can taken to be étale.

Proof. Choosing open covers of the spaces of objects consisting of linear orbifold charts, the groupoids \mathcal{G} and \mathcal{G}' each give an orbifold atlas for the orbifold Q presented by \mathcal{G} and \mathcal{G}' . These atlases need not be effective, but as they arose from a orbifold groupoids, the kernels of the actions are appropriately restricted. Let \mathcal{H} be the groupoid of the maximal atlas containing these two atlases, and then there are clearly equivalences as required. Moreover, these equivalences are strong, as the domains of charts from \mathcal{G} and \mathcal{G}' are subsets of the space of objects of \mathcal{H} so that the embeddings of these charts into the objects of \mathcal{G} and \mathcal{G}' , respectively, are surjective. \square

We solidify some notation to distinguish between the structure maps and arrows of the groupoids under consideration. We use s , t , i , u , and m to denote the source, target, inverse, unit, and composition maps of a groupoid. Often times, we will suppress m and simply express products multiplicatively by concatenation; i.e. $m(a, b) = ab$. When it is helpful to distinguish between structure maps of groupoids under consideration, we will give them subscripts of the corresponding groupoid unless otherwise indicated. For a translation groupoid $\mathcal{G} \ltimes M$, we will use the notation throughout that $s_{\mathcal{G} \ltimes M}$ and $t_{\mathcal{G} \ltimes M}$ are the source and target maps, respectively, and $(\mathcal{G} \ltimes M)_1$ is the space of arrows; note that M is the space of objects. An arrow in $(\mathcal{G} \ltimes M)_1$ is given by a $g \in G_1$ and a $z \in M$ such that the anchor map sends $s(g)$ to z . We will use (g, z) to denote this arrow so that $s_{\mathcal{G} \ltimes M}(g, z) = z$ and $t_{\mathcal{G} \ltimes M}(g, z) = gz$. In particular, for the groupoid $\mathcal{G}^\Gamma = \mathcal{G} \ltimes \mathcal{S}_\mathcal{G}^\Gamma$, an arrow is of the form (g, ϕ_x) with $s_{\mathcal{G}^\Gamma}(g, \phi_x) = \phi_x$ and $t_{\mathcal{G}^\Gamma}(g, \phi_x) = g\phi_x g^{-1}$ so that $s(g) = x$ and $t(g) = gx$.

The following lemma will simplify many of our arguments; for the definitions, see [1, Definition 2.14 and 2.15]. The proof is direct and left to the reader.

Lemma 2.6. *Let \mathcal{G} be a groupoid, and let M_1 and M_2 be \mathcal{G} -spaces with anchor maps $\alpha_i : M_i \rightarrow G_0$. Let $e_0 : M_1 \rightarrow M_2$ be a map that is \mathcal{G} -equivariant; i.e. $\alpha_2 \circ e_0 = \alpha_1$ and $e_0(hz) = he_0(z)$ for each $z \in M_1$ and $h \in G_1$ with $s(h) = \alpha_1(z)$. Define*

$$\begin{aligned} e_1 : (\mathcal{G} \ltimes M_1)_1 &\longrightarrow (\mathcal{G} \ltimes M_2)_1 \\ &: (g, z) \longmapsto (g, e_0(z)), \end{aligned}$$

and then e_0 is the map on objects and e_1 the map on arrows of a homomorphism of groupoids $e : \mathcal{G} \ltimes M_1 \rightarrow \mathcal{G} \ltimes M_2$. If e_0 is a bijection, then e is an isomorphism.

If \mathcal{G} is an orbifold groupoid, the M_i are smooth \mathcal{G} -spaces, and e_0 is smooth, then e is a homomorphism of orbifold groupoids. If e_0 is a diffeomorphism, then e is an isomorphism of orbifold groupoids.

3. CONNECTIONS BETWEEN DEFINITIONS OF SECTORS

In this section, we compare the constructions of the Γ -sectors in Section 2 with one another, as well as with other constructions of sectors in the literature.

3.1. Good Orbifold. Let Q be a good orbifold given by the quotient of a smooth manifold M by a discrete group G acting properly discontinuously. Then the translation groupoid $\mathcal{G} := G \ltimes M$ is an orbifold groupoid presenting Q , and Q admits two decompositions into Γ -sectors.

As in Subsection 2.1, we let $(M; G)_\Gamma$ denote the space of Γ -sectors of Q defined using the global G -action on M ; i.e. $(M; G)_\Gamma$ is given by $\coprod_{(\phi) \in t_{M;G}^\Gamma} C_G(\phi) \ltimes M^{(\phi)}$.

As in Subsection 2.2, we let \tilde{Q}_Γ denote the space of Γ -sectors of Q presented by $\mathcal{G}^\Gamma = \mathcal{G} \ltimes \mathcal{S}_\mathcal{G}^\Gamma$. We claim the following.

Theorem 3.1. *Let Q be a good orbifold so that $\mathcal{G} = G \ltimes M$ is an orbifold groupoid presenting Q , and let Γ be a finitely generated discrete group. Then \mathcal{G}^Γ is isomorphic as an orbifold groupoid to $G \ltimes \coprod_{\phi \in \text{Hom}(\Gamma, G)} (M^{(\phi)}, \phi)$.*

It follows that the spaces $(M; G)_\Gamma$ and \tilde{Q}_Γ are diffeomorphic as orbifolds. Before proceeding with the proof of this proposition, we note that these spaces are not indexed in the same way; the set $t_{M;G}^\Gamma$ is smaller than T_Q^Γ whenever there is a homomorphism $\phi : \Gamma \rightarrow G$ such that $\sigma(M^{(\phi)})$ is not connected.

Example 3.2. Let $\mathbb{Z}/3\mathbb{Z} = \langle \alpha \rangle$ act on S^2 by rotations; the quotient orbifold Q presented by $\mathbb{Z}/3\mathbb{Z} \ltimes S^2$ is a football with two singular points, p_s and p_n , both of which with $\mathbb{Z}/3\mathbb{Z}$ isotropy. Let $\Gamma = \mathbb{Z} = \langle \gamma \rangle$, and define

$$\begin{array}{lll} \phi_0, \phi_1, \phi_2 : \mathbb{Z} & \longrightarrow & \mathbb{Z}/3\mathbb{Z} \\ \phi_0 : \gamma & \longmapsto & 1 \\ \phi_1 : \gamma & \longmapsto & \alpha \\ \phi_2 : \gamma & \longmapsto & \alpha^2. \end{array}$$

Then the \sim -classes of the ϕ_i are the only elements of $t_{M;G}^\Gamma$. Clearly, $(M; G)_{(\phi_0)}$ is diffeomorphic to Q , and $(M; G)_{(\phi_1)}$ and $(M; G)_{(\phi_2)}$ are each diffeomorphic to $\{p_s, p_n\}$ with trivial $\mathbb{Z}/3\mathbb{Z}$ -action.

Now, consider $\mathcal{G} \ltimes \mathcal{S}_\mathcal{G}^\Gamma$. Let α_s generate G_{p_s} and α_n generate G_{p_n} for a choice of representatives of these isotropy groups. There are five \approx -classes of homomorphisms from Γ into the local groups of Q with the following representatives:

$$\begin{array}{lll} \psi_0 : \mathbb{Z} & \longrightarrow & G_p \quad \forall p \in Q \\ \psi_0 : \gamma & \longmapsto & 1 \\ \\ \psi_{1,s} : \mathbb{Z} & \longrightarrow & G_{p_s} \\ \psi_{1,s} : \gamma & \longmapsto & \alpha_s \\ \\ \psi_{2,s} : \mathbb{Z} & \longrightarrow & G_{p_s} \\ \psi_{2,s} : \gamma & \longmapsto & \alpha_s^2 \\ \\ \psi_{1,n} : \mathbb{Z} & \longrightarrow & G_{p_n} \\ \psi_{1,n} : \gamma & \longmapsto & \alpha_n \\ \\ \psi_{2,n} : \mathbb{Z} & \longrightarrow & G_{p_n} \\ \psi_{2,n} : \gamma & \longmapsto & \alpha_n^2. \end{array}$$

Then $\tilde{Q}_{(\psi_0)}$ is diffeomorphic to Q , while the sectors associated to each of the other four classes are given by a point with trivial $\mathbb{Z}/3\mathbb{Z}$ -action.

Clearly, these two decompositions result in diffeomorphic orbifolds, although the individual sectors are indexed differently.

Proof of Theorem 3.1. Let \mathcal{G} denote the translation groupoid $G \ltimes M$ so that $G_0 = M$ and $G_1 = G \times M$. Then \mathcal{G} is an orbifold groupoid in the Morita equivalence class of orbifold structures for Q . We let $\zeta : G_1 = G \times M \rightarrow G$ denote the projection onto the first factor, and then for each $\phi_x \in \mathcal{S}_{\mathcal{G}}^\Gamma$, we have $\zeta \circ \phi_x \in \text{HOM}(\Gamma, G)$; i.e.

$$\zeta \circ \phi_x : \Gamma \xrightarrow{\phi_x} (G_1)_x \xrightarrow{\zeta} G.$$

We define the map

$$\begin{aligned} \mathcal{Z} : \mathcal{S}_{\mathcal{G}}^\Gamma &\longrightarrow \coprod_{\psi \in \text{HOM}(\Gamma, G)} (M^{\langle \psi \rangle}, \psi) \\ \phi_x &\longmapsto (x, \zeta \circ \phi_x) \in (M^{\langle \zeta \circ \phi_x \rangle}, \zeta \circ \phi_x) \end{aligned}$$

Then \mathcal{Z} is clearly injective; if $\mathcal{Z}(\phi_x) = \mathcal{Z}(\psi_y)$, then $(x, \zeta \circ \phi_x) = (y, \zeta \circ \psi_y)$ so that $x = y$ and $\phi_x = \psi_y$. To show that \mathcal{Z} is surjective, let $(x, \psi) \in (M^{\langle \psi \rangle}, \psi)$ for some $\psi \in \text{HOM}(\Gamma, G)$ and define $\psi_x : \Gamma \rightarrow \mathcal{G}$ by $\psi_x(\gamma) = (\psi(\gamma), x)$. Then clearly $\mathcal{Z}(\psi_x) = (x, \psi)$, and \mathcal{Z} is a bijection. Moreover, given a chart $\kappa_{\phi_x} : V_x^{\langle \phi_x \rangle} \rightarrow \mathcal{S}_{\mathcal{G}}^\Gamma$ for $\mathcal{S}_{\mathcal{G}}^\Gamma$ near ϕ_x , we have that

$$M^{\langle \phi_x \rangle} \supseteq V_x^{\langle \phi_x \rangle} \xrightarrow{\kappa_{\phi_x}} \mathcal{S}_{\mathcal{G}}^\Gamma \xrightarrow{\mathcal{Z}} (M^{\langle \zeta \circ \phi_x \rangle}, \zeta \circ \phi_x)$$

is simply the identity on $V_x^{\langle \phi_x \rangle}$. It follows that \mathcal{Z} is smooth with smooth inverse, hence a diffeomorphism.

The anchor map of the \mathcal{G} -action on $\mathcal{S}_{\mathcal{G}}^\Gamma$ is $\beta_\Gamma : \mathcal{S}_{\mathcal{G}}^\Gamma \rightarrow M$, with $\beta_\Gamma : \phi_x \mapsto x$. Let $\alpha : \coprod_{\psi \in \text{HOM}(\Gamma, G)} (M^{\langle \psi \rangle}, \psi) \rightarrow M$ be defined by $\alpha : (x, \phi) \mapsto x$, and then α is the anchor map of a \mathcal{G} -action on $\coprod_{\psi \in \text{HOM}(\Gamma, G)} (M^{\langle \psi \rangle}, \psi) \rightarrow M$ defined by

$$(g, x)(x, \phi) = (gx, g\phi g^{-1})$$

that clearly coincides with the G -action. Hence, we need only note that for each $(g, (x, \phi)) \in G \times \coprod_{\psi \in \text{HOM}(\Gamma, G)} (M^{\langle \psi \rangle}, \psi)$ and $\phi_x \in \mathcal{S}_{\mathcal{G}}^\Gamma$ given by $\gamma \mapsto (\phi(\gamma), x)$ (so that $\mathcal{Z}(\phi_x) = (x, \phi)$),

$$\begin{aligned} (g, (x, \phi))\mathcal{Z}(\phi_x) &= (g, (x, \phi))(x, \zeta \circ \phi_x) \\ &= (g, (x, \phi))(x, \phi) \\ &= (gx, g\phi g^{-1}) \\ &= \mathcal{Z}[(g, \phi_x)\phi_x], \end{aligned}$$

and then \mathcal{Z} is \mathcal{G} -equivariant. It follows by Lemma 2.6 that \mathcal{Z} is the map on objects of an isomorphism of Lie groupoids. \square

By Lemma 2.3, we have that $G \ltimes \coprod_{\phi \in \text{HOM}(\Gamma, G)} (M^{\langle \phi \rangle}, \phi)$ and $(M; G)_\Gamma$ are Morita equivalent. Hence, by virtue of [9, Lemma 2.5] and Lemma 2.5 above, we have the following.

Corollary 3.3. *Let Q be good orbifold presented by $G \ltimes M$ with G discrete and let Γ be a finitely generated discrete group. If \mathcal{G} is any orbifold groupoid presenting Q , then $(M; G)_\Gamma$ and \mathcal{G}^Γ are Morita equivalent. Hence, the two definitions of Γ -sectors coincide. In particular, $(M; G)_\Gamma$ is Morita equivalent to an orbifold groupoid.*

Finally, we note that the proof of Theorem 3.1 generalizes readily to proper étale orbispaces. That is, we have the following.

Theorem 3.4. *Let Y be a T_1 G -space with G discrete such that the isotropy group of each point is finite, let Γ be a finitely generated discrete group, and let $\mathcal{G} = G \ltimes Y$. Then \mathcal{G}^Γ is isomorphic as a topological groupoid to $G \ltimes \coprod_{\phi \in \text{Hom}(\Gamma, G)} (M^{(\phi)}, \phi)$.*

Proof. Algebraically, the proof is identical to that of Theorem 3.1. Based on the note after Definition 2.4, the map \mathcal{Z} is clearly a homeomorphism. Moreover, the induced map on arrows given by Lemma 2.6 is clearly a homeomorphism as well. \square

3.2. Quotient Orbifolds. In the case that G is not discrete, we have the following.

Theorem 3.5. *Let G be a Lie group that acts smoothly on the smooth manifold M satisfying conditions (i), (ii), and (iii) in Subsection 2.1 so that $G \ltimes M$ presents an orbifold Q . Let \mathcal{G} be an orbifold groupoid representing Q so that $G \ltimes M$ and \mathcal{G} are Morita equivalent. Then \mathcal{G}^Γ and $(M; G)_\Gamma$ are Morita equivalent.*

Proof. First, we construct a specific orbifold groupoid that is Morita equivalent to $G \ltimes M$.

If G acts properly on M with discrete isotropy groups, then M is foliated by (connected components of) G -orbits (see [18, page 16]). Pick $x \in M$, and then there is a unique G_x -space S_x and a G -diffeomorphism of $G \times_{G_x} S_x$ onto an open subset of M containing x . We recall the construction of $G \times_{G_x} S_x$. If $(u, y) \in G \times S_x$ and $k \in G_x$, then $k(u, y) = (uk^{-1}, ky)$ defines a G_x -action on $G \times S_x$, and $G \times_{G_x} S_x$ is the orbit space of this action. Then the G -action on $G \times S_x$ given by $g'(g, y) = (g'g, y)$ induces a G -action on $G \times_{G_x} S_x$ (see [23, page 32]). In particular, the slice S_x is a transversal for the foliation of $(G \times_{G_x} S_x)$ by G -orbits. We note that S_x is not a complete transversal unless G/G_x is connected; in general, a complete transversal to the foliation of $(G \times_{G_x} S_x)$ can be formed by picking one translate gS_x of the slice of S_x in each connected component of $(G \times_{G_x} S_x)$.

As M/G is paracompact, an open cover of M/G formed by picking a chart of the form $G \times_{G_x} S_x$ for a choice of one point x in each G -orbit of M can be refined to a locally finite cover by shrinking the S_x ; hence, we can form a complete transversal S to the foliation of M by G -orbits by taking the (possibly disconnected) union of slices S_x .

By [7, Theorem 1 and Lemma 2], $G \ltimes M$ is equivalent to the groupoid given by the restriction $(G \ltimes M)|_S$ of $G \ltimes M$ to a complete transversal S (note that the *essential equivalence* of [7] corresponds to an *equivalence* in [1, Definition 1.42]; we use the language of the latter for consistency). Moreover, $(G \ltimes M)|_S$ is étale. Since $G \ltimes M$ is a proper and properness is preserved under equivalence, $(G \ltimes M)|_S$ is an orbifold groupoid.

The following argument follows [2, Theorem 5.3], which treats the case of $\Gamma = \mathbb{Z}$.

Pick a homomorphism $\phi : \Gamma \rightarrow G$ with nonempty fixed-point set in M . As G acts on M with discrete isotropy, $C_G(\phi)$ clearly acts on $M^{(\phi)}$ with discrete isotropy

and hence foliates $M^{(\phi)}$ by (connected components of) $C_G(\phi)$ -orbits. We construct a complete transversal to this foliation from the complete transversal S .

Pick a chart of the form $(G \times_{G_x} S_x)$ where the slice S_x is contained in S . Then $(G \times_{G_x} S_x)^{(\phi)}$ is by definition the set of $G_x(u, y) \in (G \times_{G_x} S_x)$ such that

$$\forall \gamma \in \Gamma \quad \exists h \in G_x : (\phi(\gamma)u, y) = h(u, y)$$

where again $h(u, y) = (uh^{-1}, hy)$. We claim that $(G \times_{G_x} S_x)^{(\phi)}$ is given by

$$(3.1) \quad \left\{ G_x(u, y) \in (G \times_{G_x} S_x) : u^{-1}(\text{Im } \phi)u \leq G_x, y \in S_x^{\langle u^{-1}\phi u \rangle} \right\}.$$

Suppose $u^{-1}(\text{Im } \phi)u \leq G_x$ and $y \in S_x^{\langle u^{-1}\phi u \rangle}$. For each $\gamma \in \Gamma$,

$$\begin{aligned} u^{-1}\phi(\gamma)^{-1}u(u, y) &= (u(u^{-1}\phi(\gamma)^{-1}u)^{-1}, u^{-1}\phi(\gamma)^{-1}uy) \\ &= (uu^{-1}\phi(\gamma)u, y) \\ &= (\phi(\gamma)u, y). \end{aligned}$$

As $u^{-1}\phi(\gamma)^{-1}u \in G_x$, it follows that the G_x -orbits $G_x(\phi(\gamma)u, y) = G_x(u, y)$. As this is true for each $\gamma \in \Gamma$, we have that $G_x(u, y) \in (S_x \times_{G_x} G)^{(\phi)}$.

Conversely, suppose the orbit $G_x(u, y)$ is fixed by $\phi(\gamma)$ for each $\gamma \in \Gamma$. Then for each $\gamma \in \Gamma$, there is an $h \in G_x$ such that $(\phi(\gamma)u, y) = h(u, y) = (uh^{-1}, hy)$. It follows that $\phi(\gamma)u = uh^{-1}$ so that $u^{-1}\phi(\gamma)u = h^{-1} \in G_x$. Moreover, $y = hy$ so that $y \in S_x^{\langle h \rangle} = S_x^{\langle u^{-1}\phi(\gamma)^{-1}u \rangle}$. As this is true for each $\gamma \in \Gamma$, $u^{-1}(\text{Im } \phi)u \leq G_x$ and $y \in S_x^{\langle u^{-1}\phi u \rangle}$, proving the expression in (3.1) of $(G \times_{G_x} S_x)^{(\phi)}$.

Now, let \mathcal{O}_ϕ be the collection of $\psi : \Gamma \rightarrow G_x \leq G$ that are conjugate to ϕ in G . Then G_x acts on $\coprod_{\psi \in \mathcal{O}_\phi} (S_x^{\langle \psi \rangle}, \psi)$ via $h(y, \psi) = (hy, h\psi h^{-1})$. We let $[y, \psi]$ denote the G_x -orbit of (y, ψ) . Define the map

$$\begin{aligned} \mathcal{E} : (G \times_{G_x} S_x)^{(\phi)} &\longrightarrow \left(\coprod_{\psi \in \mathcal{O}_\phi} (S_x^{\langle \psi \rangle}, \psi) \right) / G_x \\ : G_x(u, y) &\longmapsto [y, u^{-1}\phi u]. \end{aligned}$$

This map is well-defined, as for $h \in G_x$,

$$\begin{aligned} \mathcal{E}(G_x h(u, y)) &= \mathcal{E}(G_x(uh^{-1}, hy)) \\ &= [hy, hu^{-1}\phi uh^{-1}] \\ &= h[y, u^{-1}\phi u] \\ &= [y, u^{-1}\phi u] \\ &= \mathcal{E}(u, y). \end{aligned}$$

Note that $y \in S_x^{\langle u^{-1}\phi u \rangle}$ whenever $G_x(u, y) \in (G \times_{G_x} S_x)^{(\phi)}$, and note further that the map \mathcal{E} is clearly smooth, both observations by virtue of (3.1).

The map \mathcal{E} is not injective. However, we claim that $\mathcal{E}(G_x(u, y)) = \mathcal{E}(G_x(v, y'))$ if and only if there is a $z \in C_G(\phi)$ such that $z(G_x(u, y)) = G_x(v, y')$; i.e. $(zu, y) =$

$h(v, y') = (vh^{-1}, hy')$ for some $h \in G_x$. If this is the case, then $(u, y) = (z^{-1}vh^{-1}, hy')$, so that

$$\begin{aligned}
\mathcal{E}(G_x(u, y)) &= [y, u^{-1}\phi u] \\
&= [hy', (z^{-1}vh^{-1})^{-1}\phi z^{-1}vh^{-1}] \\
&= [hy', hv^{-1}z\phi z^{-1}vh^{-1}] \\
&= h[y', v^{-1}\phi v] \\
&= [y', v^{-1}\phi v] \\
&= \mathcal{E}(G_x(v, y')).
\end{aligned}$$

Conversely, if $\mathcal{E}(G_x(u, y)) = \mathcal{E}(G_x(v, y'))$, then $[y, u^{-1}\phi u] = [y', v^{-1}\phi v]$ so that there is an $h \in G_x$ such that $(y, u^{-1}\phi u) = h(y', v^{-1}\phi v) = (hy', hv^{-1}\phi vh^{-1})$. It follows that $y = hy'$ and $u^{-1}\phi u = hv^{-1}\phi vh^{-1}$; i.e. that $\phi = vh^{-1}u^{-1}\phi uhv^{-1}$. Hence letting $z = vh^{-1}u^{-1}$, $z \in C_G(\phi)$, and we have that $zu = vh^{-1}$, so that $(zu, y) = (vh^{-1}, hy')$.

To see that this map is surjective, let $[y, \psi] \in \coprod_{\psi \in \mathcal{O}_\phi} (S_x^{(\psi)}, \psi)$ and then there is a $u \in G$ such that $u\psi u^{-1} = \phi$. Then $(u, y) \in (G \times_{G_x} S_x)^{(\phi)}$ and $\mathcal{E}(G_x(u, y)) = [y, \psi]$.

With this, we have that \mathcal{E} induces a diffeomorphism from $(G \times_{G_x} S_x)^{(\phi)} / C_G(\phi)$ onto $(\coprod_{\psi \in \mathcal{O}_\phi} (S_x^{(\psi)}, \psi)) / G_x$. Let $(\psi)_{G_x}$ denote the G_x -conjugacy class of ψ to distinguish it from the G -conjugacy class. Recall from the proof of Lemma 2.3 that the strong equivalence

$$G_x \ltimes \coprod_{\psi \in \text{HOM}(\Gamma, G_x)} (S_x^{(\psi)}, \psi) \longrightarrow \coprod_{(\psi)_{G_x} \in \mathcal{O}_\phi / G_x} C_{G_x}(\psi) \ltimes S_x^{(\psi)}$$

restricts to an equivalence

$$G_x \ltimes \coprod_{\psi_0 \in (\psi)_{G_x}} (S_x^{(\psi)}, \psi) \longrightarrow C_{G_x}(\psi) \ltimes S_x^{(\psi)}$$

for each G_x -class $(\psi)_{G_x}$. Noting that \mathcal{O}_ϕ clearly consists of entire G_x -classes, we have that there is an equivalence

$$G_x \ltimes \coprod_{\psi \in \mathcal{O}_\phi} (S_x^{(\psi)}, \psi) \longrightarrow \coprod_{(\psi)_{G_x} \in \mathcal{O}_\phi / G_x} C_{G_x}(\psi) \ltimes S_x^{(\psi)},$$

where the G_x -action on \mathcal{O}_ϕ is by conjugation. This implies that there is a diffeomorphism

$$(3.2) \quad \left(\coprod_{\psi \in \mathcal{O}_\phi} (S_x^{(\psi)}, \psi) \right) / G_x \longrightarrow \coprod_{(\psi)_{G_x} \in \mathcal{O}_\phi / G_x} S_x^{(\psi)} / C_{G_x}(\psi).$$

Note that $(G \times_{G_x} S_x)^{(\phi)}$ is empty unless ϕ is conjugate in G to a homomorphism with image in G_x . Choose one representative ψ from each G_x -conjugacy class $(\psi)_{G_x}$. Recall that the map \mathcal{E} is constant on $C_G(\phi)$ -orbits. From its definition, \mathcal{E} maps the submanifold $S_x^{(\psi)}$ of the slice S_x to the G_x -orbit of $(S_x^{(\psi)}, \psi)$. Moreover,

if ψ is the chosen representative of the conjugacy class $(\psi)_{G_x}$, the equivalence in Lemma 2.3 maps $(S_x^{(\psi)}, \psi)$ onto $S_x^{(\psi)}$. It follows from this diffeomorphism and these observations that the disjoint union $\mathcal{S}_x = \coprod_{(\psi)_{G_x} \in \mathcal{O}_\phi/G_x} S_x^{(\psi)}$ is a complete transversal to the foliation of $(G \times_{G_x} S_x)^{(\phi)}$ by connected components of $C_G(\phi)$ -orbits. Forming \mathcal{S}_x for each chart for S as above, the (possibly disconnected) union \tilde{S} of the \mathcal{S}_x forms a complete transversal to the foliation of $M^{(\phi)}$ by the $C_G(\phi)$ -action.

As usual, let $(G \ltimes M)|_S^\Gamma$ denote the groupoid of Γ -sectors for the orbifold groupoid $(G \ltimes M)|_S$, constructed as in Subsection 2.2. Note that the space of objects of $(G \ltimes M)|_S$ is simply S while the arrows of $(G \ltimes M)|_S$ are given by $(g, x) \in G \times S$ such that $gx \in S$. Clearly, then, the isotropy group of a point $x \in S$ is simply G_x , the isotropy group of x as a point in M . It follows that the space of objects of $(G \ltimes M)|_S^\Gamma$ is the set of homomorphisms $\psi_x : \Gamma \rightarrow G_x$ for $x \in S$ with local charts given by $V_x^{(\psi_x)}$. As the action of an arrow (g, x) in $(G \ltimes M)|_S$ is given by $g\phi_x g^{-1}$, yielding a homomorphism from Γ into G_{gx} , the groupoid $(G \ltimes M)|_S^\Gamma$ is isomorphic to the restriction of the groupoid $C_G(\phi) \ltimes M^{(\phi)}$ to the complete transversal \tilde{S} given above. As this is true for each $(\phi) \in t_{M;G}^\Gamma$, it follows that there is an equivalence from $(G \ltimes M)|_S^\Gamma$ to $G \ltimes \coprod_{\phi \in \text{HOM}(\Gamma, G)} M^{(\phi)}$. Hence $(G \ltimes M)|_S^\Gamma$ and $(M; G)_\Gamma$ are Morita equivalent by Lemma 2.3.

To complete the proof, suppose that \mathcal{G} is any orbifold groupoid Morita equivalent to $G \ltimes M$. Then \mathcal{G} is Morita equivalent to $(G \ltimes M)|_S$ via étale groupoids by Lemma 2.5, implying by [9, Lemma 2.5] that the Γ -sectors of the two groupoids are Morita equivalent. □

Corollary 3.6. *Let G be a Lie group that acts smoothly on the smooth manifold M satisfying conditions (i), (ii), and (iii) in Subsection 2.1 so that $G \ltimes M$ presents an orbifold Q . Let Γ be a finitely generated discrete group. Then $(M; G)_\Gamma$ is Morita equivalent to an orbifold groupoid and hence presents an orbifold.*

While Example 3.2 illustrates that the correspondence

$$T_Q^\Gamma \ni (\phi_x)_\sim \longmapsto (\zeta \circ \phi_x)_\sim \in t_{M;G}^\Gamma$$

is not injective, it is clearly surjective. It is an obvious consequence of Theorems 3.1 and 3.5 and the fact that \sim -classes are precisely connected components of $\tilde{Q}_\Gamma = |\mathcal{G}^\Gamma|$ that each \sim -class corresponds to a connected component of a \sim -class of $(M; G)_\Gamma$.

With this, we note that the equivalence \approx defined on objects of \mathcal{G}^Γ in Subsection 2.2 can be expressed naturally on either model of $(M; G)_\Gamma$. Using the groupoid $(M; G)_\Gamma$ defined in Definition 2.1, given $(x, \phi), (y, \phi) \in M^{(\phi)}$, we say that $(x, \phi) \approx (y, \psi)$ if the orbits $C_G(\phi)x$ and $C_G(\phi)y$ are on the same connected component of $M^{(\phi)}/C_G(\phi)$. Similarly, using the Morita equivalent groupoid representing $(M; G)_\Gamma$ given by Lemma 2.3, we say that $(x, \phi) \approx (y, \psi)$ for two points $(x, \phi), (y, \psi) \in \coprod_{\psi \in \text{HOM}(\Gamma, G)} (M^{(\psi)}, \psi)$ whenever there is a $g \in G$ such that $g\phi g^{-1} = \psi$ and such that the orbits $G(x, \phi)$ and $G(y, \psi) = G(gy, \phi)$ are on the same connected component of $(\coprod_{\psi \in \text{HOM}(\Gamma, G)} (M^{(\psi)}, \psi))/G$. Clearly, the three definitions of \approx coincide in the sense that they define the same equivalence classes on the quotient space, and the \approx -classes correspond exactly to connected components. We let

$(x, \phi)_\approx$ denote the \approx -class of the point (x, ϕ) in either case and $T_{M;G}^\Gamma$ the set of \approx -classes. Then $T_{M;G}^\Gamma$ and T_Q^Γ obviously coincide.

In the same way, the definitions in [9, Section 3] can be reformulated from the perspective of a presentation as a global quotient. Let $(x, \phi)_\approx, (y, \psi)_\approx \in T_{M;G}^\Gamma$ and let $(M^{(\phi)}/C_G(\phi))_x$ and $(M^{(\psi)}/C_G(\psi))_y$ denote the connected components of $M^{(\phi)}/C_G(\phi)$ and $M^{(\psi)}/C_G(\psi)$ containing the orbits of x and y , respectively. We say that $(x, \phi)_\approx \leq (y, \psi)_\approx$ if $\pi((M^{(\phi)}/C_G(\phi))_x) \subseteq \pi((M^{(\psi)}/C_G(\psi))_y)$ where $\pi : (M; G)_\Gamma \rightarrow M/G$ denotes the map $C_G(\phi)(x, \phi) \mapsto Gx$. Similarly, Γ covers the local groups of Q if, for every $H \leq G$ such that $M^H \neq \emptyset$, there is a surjective homomorphism $\phi : \Gamma \rightarrow H$.

3.3. Connections between Γ -Sectors and Other Sectors. The definition of the Γ -sectors was motivated by that of the inertia orbifold and the k -multi-sectors given in [1, pages 52–53] (see also [6]). Hence, the Γ -sectors generalize the definition of the multi-sectors in the following sense.

Proposition 3.7. *Let Q be an orbifold presented by the orbifold groupoid \mathcal{G} and let \mathbb{F}_k denote the free group on k -generators. Then the groupoids $\mathcal{G} \ltimes \mathcal{S}_\mathcal{G}^k$ and $\mathcal{G} \ltimes \mathcal{S}_\mathcal{G}^{\mathbb{F}_k}$ are isomorphic. In particular, $\tilde{Q}_{\mathbb{F}_k}$ is diffeomorphic to the space of k -multi-sectors \tilde{Q}_k .*

Proof. This follows almost immediately from the definition. Let \mathbb{F}_k be generated by $\gamma_1, \dots, \gamma_k$, and recall from [1] that $\mathcal{S}_\mathcal{G}^k$ is defined to be the set

$$\{(g_1, \dots, g_k) : g_i \in G_1, s(g_i) = t(g_j) \quad \forall i, j \leq k\}.$$

To each $(g_1, \dots, g_k) \in \mathcal{S}_\mathcal{G}^k$ with $s(g_i) = t(g_j) = x$, there is a unique homomorphism $\phi_x : \mathbb{F}_k \rightarrow G_x$ such that $\phi_x(\gamma_i) = g_i$. It is obvious that the identification $(g_1, \dots, g_k) \mapsto \phi_x$ is a homeomorphism $\mathcal{S}_\mathcal{G}^k \rightarrow \mathcal{S}_\mathcal{G}^{\mathbb{F}_k}$. With this, we need only note that the action of \mathcal{G} on $\mathcal{S}_\mathcal{G}^k$ and $\mathcal{S}_\mathcal{G}^{\mathbb{F}_k}$ are defined identically, and hence the result follows by an application of Lemma 2.6. \square

Corollary 3.8. *Let \mathcal{G} be an orbifold groupoid. Then $\mathcal{G}^\mathbb{Z}$ is isomorphic as a groupoid to the inertia groupoid $\wedge \mathcal{G}$. In particular, the space of \mathbb{Z} -sectors $\tilde{Q}_\mathbb{Z}$ is diffeomorphic to the inertia orbifold \tilde{Q} .*

In [15], Leida defines the *fixed-point sectors* of an orbifold groupoid \mathcal{G} . Recall that Leida defines $\tilde{\mathcal{S}}(\mathcal{G}) = \{(x, H) | x \in G_0, H \leq G_x\}$, and $\tilde{\mathcal{G}} = \mathcal{G} \ltimes \tilde{\mathcal{S}}(\mathcal{G})$. Similarly, for each subgroup H of G_1 , $\tilde{\mathcal{S}}^H(\mathcal{G})$ is the subset $\{(x, K) | K \cong H\}$ given by points (x, K) where K is isomorphic to H . Define the map

$$\begin{aligned} \varrho : \mathcal{S}_\mathcal{G}^\Gamma &\longrightarrow \tilde{\mathcal{S}}(\mathcal{G}) \\ \phi_x &\longmapsto (x, \text{Im } \phi_x). \end{aligned}$$

For each point $(x, \text{Im } \phi_x) = \varrho(\phi_x)$ in the image of ϱ , there is a neighborhood V_x of x in G_0 such that the restriction $\mathcal{G}|_{V_x}$ is isomorphic to $G_x \ltimes V_x$. This corresponds to a neighborhood of $(x, \text{Im } \phi_x)$ in $\tilde{\mathcal{S}}^{\text{Im } \phi_x}(\mathcal{G})$ diffeomorphic to $V_x^{(\phi_x)}$ such that the restriction of $\tilde{\mathcal{G}}^{\text{Im } \phi_x}$ is isomorphic to $N_{G_x}(\text{Im } \phi_x) \ltimes V_x^{(\phi_x)}$ (see [15, Section 2.2]; $N_{G_x}(\text{Im } \phi_x)$ denotes the normalizer of $\text{Im } \phi_x$ in G_x). Similarly, there is a neighborhood of ϕ_x in $\mathcal{S}_\mathcal{G}^\Gamma$ such that the restriction of \mathcal{G}^Γ is isomorphic to

$C_{G_x}(\phi_x) \ltimes V_x^{\langle \phi_x \rangle}$. When restricted to these neighborhoods, the map ϱ is simply the embedding of $C_{G_x}(\phi_x) \ltimes V_x^{\langle \phi_x \rangle}$ into $N_{G_x}(\phi_x) \ltimes V_x^{\langle \phi_x \rangle}$. If ψ_x is another point with $\varrho(\psi_x) = (x, \text{Im } \phi_x)$ then $\text{Im } \psi_x = \text{Im } \phi_x$ so that $C_{G_x}(\psi_x) \ltimes V_x^{\langle \psi_x \rangle} = C_{G_x}(\phi_x) \ltimes V_x^{\langle \phi_x \rangle}$.

If a point $(x, \text{Im } \phi_x)$ is in the image of ϱ , then every point in $\tilde{\mathcal{G}}^{\text{Im } \phi_x}$ is in the image of ϱ . To see this, note that if $(y, H) \in \tilde{\mathcal{S}}^{\text{Im } \phi_x}(\mathcal{G})$, then as H is isomorphic to $\text{Im } \phi_x$, there is a homomorphism $\psi_y : \Gamma \rightarrow G_y$ with image H . It follows that $\varrho(\psi_y) = (y, H)$. Note that it need not be the case that $\psi_y \approx \phi_x$. However, using the techniques of [9, Lemma 3.2], it is easy to see that the images of Γ -sectors (ϕ) via ϱ are entire connected components of $\tilde{\mathcal{G}}$. In particular, if $(x, \text{Im } \phi_x)$ and (y, H) are in the same connected component of a fixed-point sector $\tilde{\mathcal{G}}^{\text{Im } \phi_x} = \tilde{\mathcal{G}}^H$, then they are connected by a path in $\tilde{\mathcal{G}}^H$ and hence a finite number of charts of the form $N_{G_{x_i}}(H_i) \ltimes V_{x_i}^{H_i}$ with each H_i isomorphic. Arrows $g \in G_1$ connecting orbits of point in $V_{x_i}^{H_i}$ to those in $V_{x_{i+1}}^{H_{i+1}}$ act on homomorphisms with images in H_i resulting in homomorphisms with images in H_{i+1} . Hence, a sequence of homomorphisms ϕ_{x_i} can be defined in each chart, showing that there is a $\phi_y \approx \phi_x$ with $\varrho(\phi_y) = (y, H)$.

If Γ covers the local groups of Q , i.e. for each subgroup $H \leq G_x$ of an isotropy group of Q , there is a surjective homomorphism $\Gamma \rightarrow H$, then it is clear that each (x, H) is the image via ϱ of a ϕ_x with $\text{Im } \phi_x = H$. Hence, ϱ is surjective, and we have the following.

Proposition 3.9. *Let Γ be a finitely generated discrete group and \mathcal{G} an orbifold groupoid presenting the orbifold Q . Each Γ -sector of Q is an orbifold cover of a connected component of each $\tilde{\mathcal{G}}^{\text{Im } \phi_x}(\mathcal{G})$. If ϕ_x is chosen to have minimal isotropy in (ϕ) , then the Γ -sector (ϕ) is a $[N_{G_x}(\phi_x) : C_{G_x}(\phi_x)]$ -cover of the corresponding fixed-point sector. If Γ covers the local groups of Q , then each connected component of each $\tilde{\mathcal{G}}^H$ is orbifold covered by a Γ -sector.*

As the homotopy groups of an orbifold groupoid \mathcal{G} are Morita invariant, it follows that the homotopy groups of the Γ -sectors are Morita invariants for each finitely generated discrete group Γ . See [15], [5], and [10] for more on homotopy theory of homotopy groups of orbifolds and orbispaces.

4. A MODEL OF THE Γ -SECTORS USING GENERALIZED LOOP SPACES

In [16], it is shown that the inertia orbifold of an orbifold Q occurs in the context of the loop space of Q . It appears as the subset of constant loops or equivalently the set of loops fixed by the natural S^1 -action on the loop space. In this section, we show how this construction can be generalized to demonstrate that the Γ -sectors of an orbifold arise in the same way when considering maps from a closed manifold M_Γ with fundamental group Γ . See also [21] for similar results for global quotients from a different perspective.

Many of the results in this section can be proven by direct generalizations of arguments in [16] once the appropriate definitions are given. Hence, we will be thorough with the details of the definitions and refer the reader to [16], noting any nontrivial changes. Throughout this section, we let Q be an arbitrary smooth orbifold represented by the orbifold groupoid \mathcal{G} .

4.1. The M_Γ -Multiloop Space of an Orbifold. In this subsection, we develop a groupoid structure for a manifold M_Γ with fundamental group Γ . This construction generalizes that of [16, Sections 3.1–3.2] for the case of $\Gamma = \mathbb{Z}$ and $M_\Gamma = S^1$.

Let Γ be a finitely generated discrete group, M_Γ a smooth manifold with fundamental group Γ , and M the universal cover of M_Γ so that $M/\Gamma = M_\Gamma$. We let $\pi_\Gamma : M \rightarrow M_\Gamma$ denote the covering projection. Fix a metric on M_Γ and consider a cover $\mathcal{U} = \{U_n\}_{n \in \mathbb{N}}$ of M_Γ that is $\frac{1}{n}$ -admissible; i.e. each U_n is evenly covered and has diameter $\leq \frac{1}{n}$. Note that if M_Γ is compact, we can assume that \mathcal{U} is finite. Let \mathcal{W} be the cover of M formed by the connected components of the sets $\pi_\Gamma^{-1}(U_i)$ for each $U_i \in \mathcal{U}$. In other words, for each $n \in \mathbb{N}$, choose one connected component W_n^1 of $\pi_\Gamma^{-1}(U_n)$ and let $W_n^\gamma = \gamma W_n^1$. Then $\mathcal{W} = \{W_n^\gamma\}_{n \in \mathbb{N}, \gamma \in \Gamma}$. Set $W_n = \pi_\Gamma^{-1}(U_n) = \coprod_{\gamma \in \Gamma} W_n^\gamma$, and define the groupoid $M^\mathcal{W}$ to be the groupoid associated to the covering \mathcal{W} of M . That is, the set of the units $(M^\mathcal{W})_0$ of $M^\mathcal{W}$ is given by

$$(M^\mathcal{W})_0 = \coprod_{n \in \mathbb{N}, \gamma \in \Gamma} W_n^\gamma,$$

and the set of arrows $(M^\mathcal{W})_1$ is

$$(M^\mathcal{W})_1 = \coprod_{n, m \in \mathbb{N}; \gamma, \delta \in \Gamma} W_n^\gamma \cap W_m^\delta.$$

We let (x, W_n^γ) denote the object associated to $x \in W_n^\gamma \subseteq M$ to distinguish it from (x, W_m^δ) in the case that $x \in W_n^\gamma \cap W_m^\delta$. When the specific translate of W_n^1 does not concern us, we simply use (x, W_n) . Note that this introduces no ambiguity; Γ is the group of deck translations of the manifold cover $M \rightarrow M_\Gamma$ so that x can be contained in only one translate W_n^γ of W_n^1 . Similarly, we use $W_{n,m}^{\gamma,\delta}$ to denote the connected component $W_n^\gamma \cap W_m^\delta$ of $(M^\mathcal{W})_1$ and let $W_{n,m} = \coprod_{\gamma, \delta \in \Gamma} W_{n,m}^{\gamma,\delta}$. Then $(x, W_{n,m}^{\gamma,\delta})$ or simply $(x, W_{n,m})$ (again, with no ambiguity) denotes the arrow corresponding to the point $x \in W_{n,m}^{\gamma,\delta}$. The structure maps are defined by

$$\begin{aligned} s_{M^\mathcal{W}}(x, W_{n,m}^{\gamma,\delta}) &= (x, W_n^\gamma), \\ t_{M^\mathcal{W}}(x, W_{n,m}^{\gamma,\delta}) &= (x, W_m^\delta), \\ i_{M^\mathcal{W}}(x, W_{n,m}^{\gamma,\delta}) &= (x, W_{m,n}^{\delta,\gamma}), \\ u_{M^\mathcal{W}}(x, W_n^\gamma) &= (x, W_{n,n}^{\gamma,\gamma}); \end{aligned}$$

a composable pair of arrows is of the form $((x, W_{t,n}^{\nu,\gamma}), (x, W_{n,m}^{\gamma,\delta}))$, and the composition is defined as

$$m_{M^\mathcal{W}}((x, W_{t,n}^{\nu,\gamma}), (x, W_{n,m}^{\gamma,\delta})) = (x, W_{t,m}^{\nu,\delta}).$$

Define a left Γ -action on $M^\mathcal{W}$ by

$$\Gamma \times (M^\mathcal{W})_0 \longrightarrow (M^\mathcal{W})_0$$

$$(\gamma', (x, W_n^\gamma)) \longmapsto (\gamma'x, W_n^{\gamma'\gamma})$$

and

$$\Gamma \times (M^\mathcal{W})_1 \longrightarrow (M^\mathcal{W})_1$$

$$(\gamma', (x, W_{n,m}^{\gamma,\delta})) \longmapsto (\gamma'x, W_{n,m}^{\gamma'\gamma, \gamma'\delta}).$$

The following proposition is straightforward.

Proposition 4.1. *The above is an action of the group Γ on the Lie groupoid $M^\mathcal{W}$.*

Hence, we have the following.

Definition 4.2. Let $M_\Gamma^\mathcal{W} = \Gamma \ltimes M^\mathcal{W}$ be the groupoid crossed product of Γ with the groupoid $M^\mathcal{W}$ with respect to the above action of Γ . In particular we have

$$\begin{aligned} s_{M_\Gamma^\mathcal{W}}(\gamma, (x, W_{n,m})) &= (x, W_n), \\ t_{M_\Gamma^\mathcal{W}}(\gamma, (x, W_{n,m})) &= (\gamma x, W_m), \\ i_{M_\Gamma^\mathcal{W}}(\gamma, (x, W_{n,m})) &= (\gamma^{-1}, (\gamma x, W_{m,n})), \\ u_{M_\Gamma^\mathcal{W}}(x, W_n) &= (1, (x, W_{n,n})); \end{aligned}$$

a composable pair is of the form $(\gamma, (x, W_{l,n})), (\delta, (\gamma x, W_{n,m}))$, and the composition is given by

$$m_{M_\Gamma^\mathcal{W}}[(\gamma, (x, W_{l,n})), (\delta, (\gamma x, W_{n,m}))] = (\gamma\delta, (x, W_{l,m})).$$

Proposition 4.3. The groupoid $M_\Gamma^\mathcal{W}$ is Morita equivalent to M_Γ with its trivial groupoid structure.

Proof. In fact, there is a strong equivalence from $M_\Gamma^\mathcal{W}$ to M_Γ defined on objects by $(\gamma, (x, W_n)) \mapsto \pi_\Gamma(x)$, and on arrows by mapping $(\gamma, (x, W_{n,m}))$ to the unit over $\pi_\Gamma(x)$. That this map is a strong equivalence is easy to check. \square

In the same way, one can prove the following.

Proposition 4.4. If $\tilde{\mathcal{W}}$ is a refinement of \mathcal{W} , then the natural groupoid morphism $\rho_{\tilde{\mathcal{W}}}^\mathcal{W} : M_\Gamma^{\tilde{\mathcal{W}}} \rightarrow M_\Gamma^\mathcal{W}$ is a strong equivalence.

Note that Proposition 4.3 implies that the Morita equivalence class of the groupoid $M_\Gamma^{\tilde{\mathcal{W}}}$ does not depend on the metric used to define it. More concretely using Proposition 4.4, if given two metrics on M_Γ with corresponding covers \mathcal{U}_1 and \mathcal{U}_2 (inducing covers \mathcal{W}_1 and \mathcal{W}_2 of M), one can define a strictly smaller metric and corresponding cover \mathcal{U}_3 that refines both \mathcal{U}_1 and \mathcal{U}_2 .

Definition 4.5. Let Q be a smooth orbifold presented by the orbifold groupoid \mathcal{G} , and let \mathcal{W} be a cover of M constructed from an admissible cover of M_Γ as above. The M_Γ -multiloop groupoid of \mathcal{G} corresponding to \mathcal{W} is defined to be the groupoid $\mathcal{ML}(\mathcal{W}; \mathcal{G})_{M_\Gamma}$ where

$$(\mathcal{ML}(\mathcal{W}; \mathcal{G})_{M_\Gamma})_0 = \text{HOM}(M_{M_\Gamma}^\mathcal{W}, \mathcal{G})$$

is the set of Lie groupoid homomorphisms from $M_\Gamma^\mathcal{W}$ to \mathcal{G} . The arrows in $\mathcal{ML}(\mathcal{W}; \mathcal{G})_{M_\Gamma}$ are defined as follows. For any two elements $\Phi, \Psi \in \text{HOM}(M_\Gamma^\mathcal{W}, \mathcal{G})$, an arrow from Ψ to Φ is a map $\Lambda : (M_\Gamma^\mathcal{W})_1 \rightarrow (\mathcal{G})_1$ such that the following diagram commutes

$$\begin{array}{ccc} (M_\Gamma^\mathcal{W})_1 & \xrightarrow{\Lambda} & G_1 \\ s_{M_\Gamma^\mathcal{W}} \times t_{M_\Gamma^\mathcal{W}} \downarrow & & \downarrow s \times t \\ (M_\Gamma^\mathcal{W})_0 \times (M_\Gamma^\mathcal{W})_0 & \xrightarrow{\Psi_0 \times \Phi_0} & G_0 \times G_0 \end{array}$$

and such that for every $(\gamma, (x, W_n)) \in (M_\Gamma^\mathcal{W})_1$, we have

$$\begin{aligned}
 \Lambda(\gamma, (x, W_n)) &= \Psi_1(\gamma, (x, W_n)) \Lambda \left[u_{M_\Gamma^\mathcal{W}} \circ s_{M_\Gamma^\mathcal{W}}(\gamma, (x, W_n)) \right] \\
 (4.1) \qquad &= \Lambda \left[u_{M_\Gamma^\mathcal{W}} \circ t_{M_\Gamma^\mathcal{W}}(\gamma, (x, W_n)) \right] \Phi_1(\gamma, (x, W_n))
 \end{aligned}$$

where as usual Φ_1 and Ψ_1 denote the maps on arrows given by Φ and Ψ , respectively. Note that the above product is taken in G_1 so that the target of the right element is equal to the source of the left.

If $\Lambda : \Psi \rightarrow \Phi$ and $\Omega : \Phi \rightarrow \Xi$, then the composition $\Omega \circ \Lambda$ is defined by

$$\begin{aligned}
 \Omega \circ \Lambda \left[u_{M_\Gamma^\mathcal{W}} \circ t_{M_\Gamma^\mathcal{W}}(\gamma, (x, W_n)) \right] \\
 = \Lambda \left[u_{M_\Gamma^\mathcal{W}} \circ t_{M_\Gamma^\mathcal{W}}(\gamma, (x, W_n)) \right] \Omega \left[u_{M_\Gamma^\mathcal{W}} \circ t_{M_\Gamma^\mathcal{W}}(\gamma, (x, W_n)) \right]
 \end{aligned}$$

and

$$\Omega \circ \Lambda(\gamma, (x, W_n)) = \Omega \circ \Lambda \left[u_{M_\Gamma^\mathcal{W}} \circ t_{M_\Gamma^\mathcal{W}}(\gamma, (x, W_n)) \right] \Xi_1(\gamma, (x, W_n))$$

Under the compact-open topology, $\mathcal{ML}(\mathcal{W}; \mathcal{G})_{M_\Gamma}$ is a topological groupoid. Note that for each arrow in $\Lambda \in \mathcal{ML}(\mathcal{W}; \mathcal{G})_{M_\Gamma}$ from Φ to Ψ , $\Lambda \circ u_{M_{M_\Gamma}^\mathcal{W}}$ is a natural transformation from Φ to Ψ (see [1, Definition 1.40]).

Compare the following to [16, Definition 3.2.2].

Definition 4.6. Let Q be a smooth orbifold presented by the orbifold groupoid \mathcal{G} . The M_Γ -multiloop groupoid of \mathcal{G} denoted $\mathcal{ML}(\mathcal{G})_{M_\Gamma}$ is the colimit of the $\mathcal{ML}(\mathcal{W}; \mathcal{G})_{M_\Gamma}$ over all admissible covers of M_Γ partially ordered by inclusion of cover charts.

In order to show that the M_Γ -multiloop groupoid $\mathcal{ML}(\mathcal{G})_{M_\Gamma}$ of an orbifold Q presented by the orbifold groupoid \mathcal{G} is étale, we first have the following.

Lemma 4.7. Let $\mathcal{ML}(\mathcal{G})_{M_\Gamma}$ be the M_Γ -multiloop groupoid of an orbifold Q presented by the orbifold groupoid \mathcal{G} . Then any arrow $\Lambda : \Psi \rightarrow \Phi$ is completely determined by Ψ and by $\Lambda \circ u_{M_\Gamma^\mathcal{W}}(x, W_n)$ for any $(x, W_n) \in (M_\Gamma^\mathcal{W})_0$.

Proof. Straightforward from the definitions; see [16, Lemma 3.2.4]. On any given W_n , Λ is determined by Ψ , a single value $\Lambda \circ u_{M_\Gamma^\mathcal{W}}(x, W_n)$, and Equation 4.1. This determines Λ on each chart W_m such that $W_n \cap W_m \neq \emptyset$, and hence recursively on every chart. □

Proposition 4.8. Let $\mathcal{ML}(\mathcal{G})_{M_\Gamma}$ be the M_Γ -multiloop groupoid of an orbifold Q presented by the orbifold groupoid \mathcal{G} . Then $\mathcal{ML}(\mathcal{G})_{M_\Gamma}$ is étale.

Proof. Again straightforward from the definitions. An arrow $\Lambda : \Psi \rightarrow \Phi$ is determined by Ψ and a single value $\Lambda \circ u_{M_\Gamma^\mathcal{W}}(x, W_n)$. The result then follows from the fact that the isotropy groups of \mathcal{G} are finite. □

The next result follows [16, Section 3.4].

Proposition 4.9. *Let $\mathcal{ML}(\mathcal{G}_i)_{M_\Gamma}$ be the M_Γ -multiloop groupoid of orbifolds Q_i presented by the orbifold groupoids \mathcal{G}_i for $i = 1, 2$. A groupoid homomorphism $e : Q_1 \rightarrow Q_2$ induces a homomorphism of M_Γ -multiloop groupoids $e_{\mathcal{ML}} : \mathcal{ML}(\mathcal{G}_1)_{M_\Gamma} \rightarrow \mathcal{ML}(\mathcal{G}_2)_{M_\Gamma}$. If e is a strong equivalence, then $e_{\mathcal{ML}}$ is a strong equivalence.*

Proof. The proof is identical to that of [16, Definition 3.4.1 and Lemma 3.4.2]. \square

4.2. The M_Γ -Multiloops when Γ is a Subgroup of a Contractible Abelian Group. In this subsection, we assume that Γ is a subgroup of a contractible abelian Lie group T ; in the notation of Subsection 4.1, $T = M$ and $T/\Gamma = M_\Gamma$. Following [16, Section 3.6], we recover the Γ -sectors of Q from the fixed points of the T/Γ -multiloop groupoid. First, we define a T -action on the T/Γ -multiloop groupoid $\mathcal{ML}(\mathcal{G})_{T/\Gamma}$.

Definition 4.10. *Suppose Γ is a subgroup of a contractible abelian Lie group T . Let $\mathcal{ML}(\mathcal{W}; \mathcal{G})_{T/\Gamma}$ be the T/Γ -multiloop groupoid associated to the cover \mathcal{W} of T given by Definition 4.5. For each $W_n = \coprod_{\gamma \in \Gamma} W_n^\gamma$ and $t \in T$, let W_n^t denote the translate $tW_n = \{tx : x \in W_n\}$, and let \mathcal{W}^t denote the translated cover $\{W_n^t\}_{n \in \mathbb{N}}$. Note that this introduces no ambiguity; W_n^t has the same meaning as in Subsection 4.1 when $t \in \Gamma \leq T$. Then T acts on $\coprod_{t \in T} \mathcal{W}^t$ via $(s, (x, W_n^t)) \mapsto (sx, W_n^{st})$ for $s \in T$. As T is abelian, this action descends to a T -action on the cover $\coprod_{n \in \mathbb{N}, t \in T/\Gamma} U_n^{t\Gamma}$ of T/Γ -translates of \mathcal{U} in the same way.*

Now define an action of T on $\coprod_{t \in T} \mathcal{ML}(\mathcal{W}^t; \mathcal{G})_{T/\Gamma}$ by

$$\begin{aligned} T \times \left(\coprod_{t \in T} \mathcal{ML}(\mathcal{W}^t; \mathcal{G})_{T/\Gamma} \right)_0 &\longrightarrow \left(\coprod_{t \in T} \mathcal{ML}(\mathcal{W}^t; \mathcal{G})_{T/\Gamma} \right)_0 \\ (t, \Psi) &\longmapsto \Psi^t \end{aligned}$$

where Ψ^t is defined by

$$\Psi_0^t(x, W_n^t) = \Psi(t^{-1}x, W_n),$$

$$\Psi_1^t(\gamma, (x, W_{n,m}^t)) = \Psi_1(\gamma, (t^{-1}x, W_{n,m})).$$

Taking the colimit, we obtain an action of T on $\mathcal{ML}(\mathcal{G})_{T/\Gamma}$.

Now consider the subgroupoid $\mathcal{ML}(\mathcal{G})_{T/\Gamma}^T$ of $\mathcal{ML}(\mathcal{G})_{T/\Gamma}$ consisting of elements fixed by the action of T . In Theorem 4.13 below, we will show that $\mathcal{ML}(\mathcal{G})_{T/\Gamma}^T$ is Morita equivalent to \mathcal{G}^Γ , the groupoid presenting the Γ -sectors of Q . When $\Gamma = \mathbb{Z}$, this coincides with [16, Theorem 3.6.4 and Proposition 3.6.6]; see also [8]. The following two lemmas can be proved in the same way as in [16, Lemmas 3.6.2 and 3.6.3]. We give the proof of Lemma 4.12 explicitly, because it is important for the proof of Theorem 4.13.

Lemma 4.11. *For any object Ψ of $\mathcal{ML}(\mathcal{G})_{T/\Gamma}^T$, Ψ_0 and Ψ_1 are locally constant.*

Lemma 4.12. *For any object Ψ of $\mathcal{ML}(\mathcal{G})_{T/\Gamma}^T$, there is another object Φ of $\mathcal{ML}(\mathcal{G})_{T/\Gamma}^T$ defined over the trivial cover of T by one chart such that there is an arrow Λ connecting Ψ and Φ .*

Proof. Given a point $(x, W) \in (T^{\mathcal{W}})_0$, if $(y, W') \in (T^{\mathcal{W}})_0$ is another point in $(T^{\mathcal{W}})_0$, then there exists a finite collection W^1, \dots, W^r of sets in the cover \mathcal{W} such that $W^1 = W$, $W^r = W'$, and $W^i \cap W^{i+1} \neq \emptyset$ in T for $i = 1, 2, \dots, r-1$.

For each (z_i, W^i) such that $z_i \in W^i \cap W^{i+1}$, we have that $\Psi_1(1, (z_i, W^i \cap W^{i+1}))$ is an arrow from $\Psi_0(z_i, W^i)$ to $\Psi_0(z_i, W^{i+1})$. We define the arrow $A_i^{i+1}(z_i) \in (T^{\mathcal{W}})_1$ from $\Psi_0(z_i, W^i)$ to $\Psi_0(z_i, W^{i+1})$ by

$$A_i^{i+1}(z_i) = \Psi_1(1, (z_i, W^i \cap W^{i+1}))$$

and note that as Ψ_0 and Ψ_1 are locally constant by Lemma 4.11, A_i^{i+1} does not depend on z_i . Choosing one $z_i \in W^i \cap W^{i+1}$ for each i , we set

$$A_W^{W'} = \Psi_1(1, (z_{r-1}, W^{r-1} \cap W^r)) \Psi_1(1, (z_{r-2}, W^{r-2} \cap W^{r-1})) \cdots \Psi_1(1, (z_1, W^1 \cap W^2)).$$

Note that the definition of $A_W^{W'}$ depends only on the sets W and W' , and that $(A_W^{W'})^{-1} = A_{W'}^W$.

Define a morphism $\Phi : T_{\Gamma}^{\mathcal{W}} \rightarrow \mathcal{G}$ by

$$\Phi_0(z, W_n) = \Psi_0(x, W)$$

$$\Phi_1(\gamma, (z, W_{n,m})) = A_{W_n}^{W_m} \Psi_1(\gamma, (z, W_{m,n})) A_{W_n}^{W_m}.$$

Because Ψ_1 is locally constant,

$$\Phi_1(1, (z, W_{n,m})) = \Psi_1(1, (z, W_{n,m})),$$

and

$$\Phi_1(\gamma, (z, W_{n,m})) = \Psi_1(\gamma, (z, W_{n,m}))$$

for each $z \in W_{n,m}$. Hence we can define Φ on the trivial cover of T consisting of points (z, T) by

$$\Phi_0(z, T) = \Phi_0(x, W)$$

$$\Phi_1(\gamma, (z, T)) = \Phi_1(\gamma, (z, W_{n,m}))$$

whenever $z \in W_n \cap W_m$. Lastly, we define an arrow $\Lambda : \Psi \rightarrow \Phi$ by $\Lambda(1, (z, W_{n,m})) = A_{W_n}^{W_m}$.

□

We now have the following.

Theorem 4.13. *There is a strong equivalence from $\mathcal{ML}(\mathcal{G})_{T/\Gamma}^T$ to the groupoid \mathcal{G}^{Γ} of Γ -sectors of Q .*

Proof. See the proof of [16, Theorem 3.6.4]. Given $\Psi \in \mathcal{ML}(\mathcal{G})_{T/\Gamma}^T$, let Φ be as in the proof of Lemma 4.12. As Φ_0 is locally constant, $\Phi_0(y, T) = \Phi_0(1, T)$ for each $y \in T$. We have that

$$\begin{aligned} s \circ \Phi_1(\gamma, (y, T)) &= \Phi_0(y, T) \\ &= \Phi_0(\gamma y, T) \\ &= t \circ \Phi_1(\gamma, (y, T)) \end{aligned}$$

so that each $\Phi_1(\gamma, (y, T))$ is an element of the isotropy group $G_{\Phi_0(y)} = G_{\Phi_0(1)}$. Hence, we can define a homomorphism $\phi : \Gamma \rightarrow G_{\Phi_0(1)}$ by

$$\phi(\gamma) = \Phi_1(\gamma, (1, T)).$$

Clearly, the correspondence $\Psi \mapsto \phi$ is surjective, as given any $\phi_x : \Gamma \rightarrow G_x$, one can define a $\Psi \in \mathcal{ML}(\mathcal{G})_{T/\Gamma}^T$ with $\Psi_0(y, T) = x$ and $\Psi_1(\gamma, (y, T)) = \phi_x(\gamma)$. That this correspondence is a strong equivalence of groupoids is straightforward. \square

It follows that $\mathcal{ML}(\mathcal{G})_{T/\Gamma}^T$ is Morita equivalent to \mathcal{G}^Γ .

Note that in [5, Proposition 3.5.3], Chen proves that the in the case of a proper étale topological groupoid \mathcal{G} representing an orbispace such that the space of objects is a T_1 space, there is an identification similar to that given by Theorem 4.13 on the level of orbispaces for the case $\Gamma = \mathbb{Z}$. Using exactly the same proof with the definitions given above and Equation 2.1, we have the following.

Proposition 4.14. *Let X be an étale proper orbispace, that is, an orbispace represented by the étale proper groupoid \mathcal{G} , such that G_0 is a T_1 space. Let Γ be a discrete subgroup of a contractible abelian Lie group T . Then the orbit space $|\mathcal{ML}(\mathcal{G})_{T/\Gamma}^T|$ of the groupoid $\mathcal{ML}(\mathcal{G})_{T/\Gamma}^T$ is homeomorphic to $|\mathcal{G}^\Gamma|$.*

4.3. The M_Γ -Multiloop and the Γ -Sectors in the General Case. In the general case of M_Γ an arbitrary manifold with fundamental group Γ and universal cover Γ , we have a correspondence similar to Theorem 4.13. In this case, we use the groupoid of constants, a subgroupoid of $\mathcal{ML}(\mathcal{G})_{M_\Gamma}$.

Definition 4.15. *Let Q be an orbifold presented by the orbifold groupoid \mathcal{G} . The groupoid of constants $\mathcal{C}(\mathcal{G})_{M_\Gamma}$ of $\mathcal{ML}(\mathcal{G})_{M_\Gamma}$ is defined to be the subgroupoid of $\mathcal{ML}(\mathcal{G})_{M_\Gamma}$ consisting of the Φ such that $\sigma \circ \Phi$ is constant. Recall that $\sigma : \mathcal{G} \rightarrow |\mathcal{G}|$ denotes the quotient map onto the orbit space of \mathcal{G} .*

Theorem 4.16. *There is a strong equivalence from the groupoid of constants $\mathcal{C}(\mathcal{G})_{M_\Gamma}$ to the groupoid \mathcal{G}^Γ of Γ -sectors of Q .*

The proof is identical to that of Theorem 4.13.

4.4. The M_Γ -Multiloops of a Quotient Orbifold. In this subsection, we specialize to the case where M_Γ is compact and Q is presented by as the quotient of a smooth connected manifold X by a compact Lie group G acting locally freely (i.e. properly with discrete stabilizers). In the case of G finite, a very explicit characterization of the loop space is given in [16, Section 4.1]. It is shown that it is enough to consider only the homomorphisms defined on the trivial cover. Here, we briefly explain how this characterization extends readily to the case of G compact and the M_Γ -multiloops. Throughout this section, we let \mathcal{G} denote an orbifold groupoid Morita equivalent to $G \ltimes X$. In particular, we can take \mathcal{G} to be given by a collection of slices for the G -action as in Theorem 3.5.

First, we note the following. The proof is similar to that of Lemma 4.12 and, given the modifications outlined in Definitions 4.2, 4.5, and 4.15 and the local structure of quotient orbifolds demonstrated by Theorem 3.5, identical to that of [16, Lemma 4.1.1].

Proposition 4.17. *Let Q be a quotient orbifold presented by $G \ltimes X$ with G a compact Lie group acting locally freely on the smooth manifold X . Let $\mathcal{ML}(\mathcal{G})_{M_\Gamma}$ be the M_Γ -multiloops for M_Γ compact. Then for any morphism $\Psi : M_\Gamma^\mathcal{W} \rightarrow \mathcal{G}$, there is a morphism $\Phi : M_\Gamma^{\{M\}} \rightarrow \mathcal{G}$ subordinate to the trivial cover of M_Γ by M and an arrow connecting Ψ to Φ .*

Similarly, the morphisms Ψ subordinated to the trivial cover of M_Γ is determined by the image of $\coprod_{i=1}^s M \times \{\gamma_i\}$ under Ψ_1 where $\{\gamma_1, \dots, \gamma_s\}$ is a set of generators of Γ ; compare [16, Section 3.3].

Lemma 4.18. *Let Q be a quotient orbifold presented by $G \ltimes X$ with G a compact Lie group acting locally freely on the smooth manifold X . Every morphism in $\Psi \in \mathcal{ML}(\mathcal{G})_{M_\Gamma}$ subordinated to the trivial cover of M_Γ is determined by the image of $\coprod_{i=1}^n M \times \{g_i\}$ under Ψ_1 , where $\{g_1, \dots, g_n\}$ is a set of generators of Γ .*

Proof. Pick a set of generators $\{\gamma_1, \dots, \gamma_s\}$ of Γ and let $(x, \gamma) \in \Gamma \ltimes M$. Then if $\gamma = \gamma_{\alpha_1}^{\beta_1} \dots \gamma_{\alpha_s}^{\beta_s}$ is an expression of γ in terms of these generators,

$$(x, \gamma) = (x, g_{\alpha_1})(g_{\alpha_1}x, g_{\alpha_1}^{\beta_1-1}) \dots (g_{\alpha_s}^{\beta_s-1} \dots g_{\alpha_1}^{\beta_1}x, g_{\alpha_s}).$$

It follows that

$$\Psi_1(x, \gamma) = \Psi_1(x, g_{\alpha_1})\Psi_1(x, g_{\alpha_1}^{\beta_1-1}) \dots \Psi_1(x, g_{\alpha_s}^{\beta_s-1} \dots g_{\alpha_1}^{\beta_1}).$$

□

Fixing a generating set $\{\gamma_1, \dots, \gamma_s\}$ of Γ , it follows that there is a bijective correspondence between the morphisms Ψ subordinated to the trivial cover of M_Γ in $\mathcal{ML}(\mathcal{G})_\Gamma$ and the collection of pairs (f, Θ) where $\Theta = \{g_1, \dots, g_n\}$ is an s -tuple of elements of G satisfying the same relations as the γ_i , and $f : M \rightarrow X$ is a smooth map such that $g_i f(x) = f(\gamma_i x)$ for each $i = 1, \dots, s$. Let \mathcal{P}_Θ denote the set of all such pairs. Similarly, let Λ be an arrow between homomorphisms $\Psi = (f, \Theta)$ and $\Phi = (f', \Upsilon)$ with $\Theta = \{g_1, \dots, g_s\}$ and $\Upsilon = \{k_1, \dots, k_s\}$. Using the fact that X is connected, there is an $h \in G$ such that $k_i = h g_i h^{-1}$ and $h f(x) = f'(x)$ for each $i = 1, \dots, s$ and $x \in M$. Thus we have the following consequence of Lemma 4.7.

Proposition 4.19. *Let G act on \mathcal{P}_Θ via*

$$[h, (f, \Theta)] \longmapsto (hf, h\Theta h^{-1}),$$

where $h\Theta h^{-1}$ indicates pointwise conjugation of the s -tuple Θ . The crossed product groupoid $G \ltimes \mathcal{P}_\Theta$ is Morita equivalent to the orbifold M_Γ -multiloop groupoid $\mathcal{ML}(\mathcal{G})_{M^\Gamma}$ of Q .

For each s -tuple $\Theta = (g_1, \dots, g_s)$, let $C_G(\Theta)$ denote the centralizer of the subgroup generated by the g_i . Techniques identical to those in Lemma 2.3 demonstrate that the crossed-product groupoid $G \ltimes \mathcal{P}_\Theta$ is given Morita equivalent to

$$\coprod_{(\Theta)} (C_G(\Theta) \ltimes \mathcal{P}_\Theta),$$

where the sum is over the G -conjugacy classes (Θ) of the s -tuples Θ . Hence we have the following.

Corollary 4.20. *The groupoid*

$$\coprod_{(\Theta)} (C_G(\Theta) \ltimes \mathcal{P}_\Theta),$$

is Morita equivalent to the orbifold M_Γ -multiloop groupoid $\mathcal{ML}(\mathcal{G})_{M^\Gamma}$ of Q .

Note that in [21, Equation 2-12, page 808], the multiloop Γ -sectors for global quotients are given by

$$\mathbb{L}_{M_\Gamma}(X; G) = \coprod_{(\phi)} \text{Map}_\phi(M; X)/C_G(\phi)$$

where the space Map_ϕ is defined as

$$\text{Map}_\phi(M; X) = \{f : M \rightarrow X \mid f(\gamma x) = \phi(\gamma)f(x) \ \forall x \in M, \gamma \in \Gamma\}$$

(with notation modified to our case, and the Γ -action on M expressed as a left action). Hence \mathbb{L}_{M_Γ} coincides with the groupoid in Corollary 4.20 in the case of G finite.

Restricting to the groupoid of constant maps (f, Θ) in $\mathcal{ML}(G)_{M_\Gamma}$ as in Definition 4.15, we have that if $\Theta = (g_1, \dots, g_s)$, then

$$g_i f(x) = f(\gamma_i x) = f(x)$$

for each $x \in X$ and $i = 1, \dots, s$. Hence the image of f is fixed by each g_i .

Corollary 4.21. *The subgroupoid of constants $CL(\mathcal{G})_{M_\Gamma}$ of $\mathcal{ML}(G)_\Gamma$ is given by*

$$\coprod_{(\Theta)} \left(C_G(\Theta) \ltimes X^{(\Theta)} \right),$$

where the sum is over the G -conjugacy classes (Θ) of the s -tuples Θ , and is Morita equivalent to the groupoid \mathcal{G}^Γ presenting the Γ -sectors \tilde{Q}_Γ .

REFERENCES

1. A. Adem, J. Leida, and Y. Ruan. *Orbifolds and stringy topology*, Cambridge Tracts in Mathematics 171, Cambridge University Press, Cambridge, 2007.
2. A. Adem and Y. Ruan, *Twisted orbifold K-theory*, Commun. Math. Phys. 237 (2003) 533–556.
3. M. Boileau, S. Maillot, and J. Porti, *Three-dimensional orbifolds and their geometric structures*, Panoramas et Synthèses, 15. Socit Mathématique de France, Paris, 2003.
4. J. Bryan and J. Fulman, *Orbifold Euler characteristics and the number of commuting m -tuples in the symmetric groups*, Ann. Comb. 2 (1998) 1–6.
5. W. Chen, *On a notion of maps between orbifolds II: Homotopy and CW-complex*, Commun. Contemp. Math. 8 (2006) 763–821.
6. W. Chen and Y. Ruan, *A new cohomology theory of orbifold*, Commun. Math. Phys. 248 (2004) 1–31.
7. M. Crainic and I. Moerdijk, *Foliation groupoids and their cyclic homology*, Adv. Math. 157 (2001) 177–197.
8. T. de Fernet, E. Lupercio and B. Uribe, *A localization principle for orbifold theories*, in: Recent developments in algebraic topology, Contemp. Math. 407, 113–133, Amer. Math. Soc., Providence, 2006.
9. C. Farsi and C. Seaton, *Nonvanishing vector fields on orbifolds*, to appear in Trans. Amer. Math. Soc., [arXiv:0807.2738v1 \[math.DG\]](#) (2008).
10. A. Henriques and D. Gepner, *Homotopy theory of orbispaces*, preprint, [arXiv:math/0701916v1 \[math.AT\]](#), (2007).
11. Henriques, A. and D. Metzler, *Presentations of noneffective orbifolds*, Trans. Amer. Math. Soc. 356 (2004) 2481–2499.
12. Kawasaki, T. *The signature theorem for V-manifolds*, Topology 17 (1978) 75–83.
13. Kawasaki, T. *The Riemann-Roch theorem for complex V-manifolds*, Osaka J. Math. 16 (1979) 151–159.
14. Kawasaki, T. *The index of elliptic operators over V-manifolds*, Nagoya Math J. 84 (1981) 135–157.
15. Leida, J. *Orbifolds and stable homotopy groups*, preprint. [arXiv:math/0505431v1 \[math.AT\]](#) (2005).

16. Lupercio, E. and B. Uribe, *Loop groupoids, gerbes, and twisted sectors on orbifolds*, in: Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math. 310, 163–184, Amer. Math. Soc., Providence, RI, 2002.
17. I. Moerdijk, *Orbifolds as groupoids: an introduction*, in: Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math. 310, 205–222, Amer. Math. Soc., Providence, RI, 2002.
18. I. Moerdijk and J. Mrčun, *Introduction to foliations and Lie groupoids*, Cambridge studies in advanced mathematics 91, Cambridge University Press, Cambridge, NY, 2003.
19. Y. Ruan, *Stringy geometry and topology of orbifolds*, in: Symposium in Honor of C. H. Clemens (Salt Lake City, UT, 2000), Contemp. Math. 312, 187–233, Amer. Math. Soc., Providence, RI, 2002.
20. H. Tamanoi, *Generalized orbifold Euler characteristic of symmetric products and equivariant Morava K-theory*, Algebraic and Geometric Topology 1 (2001) 115–141.
21. H. Tamanoi, *Generalized orbifold Euler characteristic of symmetric orbifolds and covering spaces*, Algebraic and Geometric Topology 3 (2003) 791–856.
22. W. Thurston, *The geometry and topology of 3-manifolds*, Lecture Notes, Princeton University Math Dept., Princeton, New Jersey, 1978.
23. T. tom Dieck, *Transformation groups*, de Gruyter Studies in Mathematics 8, Walter de Gruyter, Berlin, 1987.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLORADO AT BOULDER, CAMPUS BOX 395,
BOULDER, CO 80309-0395

E-mail address: `farsi@euclid.colorado.edu`

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, RHODES COLLEGE, 2000 N. PARK-
WAY, MEMPHIS, TN 38112

E-mail address: `seatonc@rhodes.edu`